Test 3 Numerical Mathematics 2
January, 2020

Duration: 1.0 hour.

In front of the questions one finds the points. The sum of the points plus 1 gives the end mark for this test.

1. [2] Given the inner product $(f,g) = \int_{-\infty}^{\infty} \exp(-x^2) f(x) g(x) dx$, give the associated first three orthogonal polynomials; so upto degree 2. You do not need to normalize the polynomials.

2. (a) [1.5] For polynomial interpolation of functions it holds that $E_{n,\infty} \leq (1+\Lambda_n(X))E_n^*$, where $E_{n,\infty}$ is the interpolation error and E_n^* is the minimax error, i.e. the error one obtains for the best polynomial approximation of the function at hand. Furthermore, $\Lambda_n(X)$ is the Lebesgue constant. Derive this relation.

   (b) [1] Give the expression for the interpolation error and explain that taking the zeros of a Chebyshev polynomial as interpolation points is a reasonable choice.

3. Suppose the least squares approximation of a function $f(x)$ is given by $\sum_{i=0}^{\infty} \alpha_i \phi_i(x)$ where $\phi_i(x)$ are orthogonal polynomials with respect to some innerproduct $(f,g) = \int_a^b f(x)g(x)dx$.

   (a) [1.5] Determine α_i.

   (b) [0.5] Consider the partial sum of the approximation of f: $F_n(x) = \sum_{i=0}^{n} \alpha_i \phi_i(x)$. Explain the difference between convergence in the norm associated to the above inner product and pointwise convergence.

   (c) [0.5] Suppose $F_n(x)$ defined in the previous part converges pointwise to $f(x)$. Will the derivative of $F_n(x)$ also converge pointwise to the derivative of $f(x)$?

4. [2] Explain how orthogonal polynomials can be used to construct a method to solve the initial value problem
$$\frac{dy}{dt} = f(t,y), \quad \text{with } y(0) = y_0.$$

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$(f,g) = \int_{-\infty}^{\infty} \exp(-x^2) \, f g \, dx$ $\qquad \varphi_0 \equiv 1$

$(x,1) = \int_{-\infty}^{\infty} \underbrace{x \exp(-x^2)}_{\text{odd}} dx = 0 \qquad \varphi_1 = x$

$\varphi_2(x) = x^2 - \alpha \varphi_0 - \beta \varphi_1(x)$

$(\varphi_0, \varphi_2) = 0 \quad (\varphi_0, x^2) - \alpha(\varphi_0, \varphi_0) - \beta\cancel{(\varphi_0, \varphi_1)} = 0$

$\alpha = \frac{(\varphi_0, x^2)}{(\varphi_0, \varphi_0)} = \frac{1}{2}$

$(\varphi_0, x^2) = \int_{-\infty}^{\infty} x^2 \exp(-x^2) \, dx = -\frac{1}{2}\int_{-\infty}^{\infty} x \, d\exp(-x^2) =$

$-\frac{1}{2} x \exp(-x^2)\Big|_{-\infty}^{\infty} + \frac{1}{2} \underbrace{\int_{-\infty}^{\infty} \exp(-x^2) \, dx}_{\text{nasty}} = \frac{1}{2}(\varphi_0, \varphi_0)$

$(\varphi_1, \varphi_2) = 0$

$\underbrace{(x, x^2)}_{=0} - \alpha \underbrace{(x, 1)}_{0} - \beta(x, x) = 0 \quad \to \beta = 0 \qquad \varphi_2 = x^2 - \frac{1}{2}$

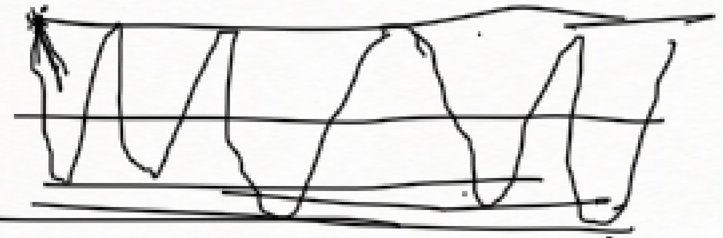
$$f(x) - p_n(x) = \underbrace{(x-x_0)(x-x_1)\cdots(x-x_n)}\,\frac{f^{(n+1)}(\xi)}{(n+1)!}$$

becomes a scaled Chebishev pol.

if $x_0 \dots x_n$ are zero's of the ~~tn+1~~ $T_{n+1}(x)$

$T_{n+1}(x)$ satisfy minmax property: all extrema are equal in magnitude and alternating

$T_{n+1}(\cos\theta) = \cos((n+1)\theta)$



$f(x) \approx \sum_{i=0}^{n} \alpha_i \varphi_i(x)$ L.S. $\min_{\alpha_k} \| f(x) - \sum_{i=0}^{n} \alpha_i \varphi_i(x) \|_2$

$\min_{\alpha_k} (f(x) - \Sigma, f(x) - \Sigma) \approx$

$\frac{\partial}{\partial \alpha_k} (\; , \;) = 0 \longrightarrow \left(-\varphi_k(x), f(x) - \sum_{i=0}^{n} \alpha_i \varphi_i(x) \right) = 0$

$\alpha_k (\varphi_k, \varphi_k) = (\varphi_k, f) \to \alpha_k = \frac{(\varphi_k, f)}{(\varphi_k, \varphi_k)}$

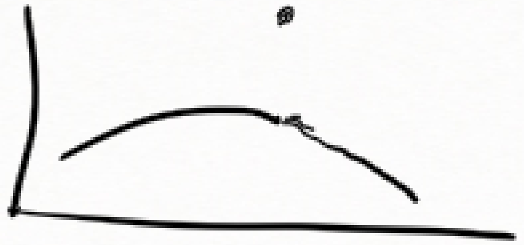
$F_n(x)$

Pointwise convergence : $F_n(x) \to f(x)$ for $n \to \infty$
for each x in $[a, b]$

Converge in L_2 norm : $\| F_n(x) - f(x) \|_2 \to 0$ for $n \to \infty$

This is weaker than pointwise convergence

$f(x)$ may have points x_i which have different values $f(x_i)$ from $f(x_i + \varepsilon)$



$F_n(x)$ is a least square approxim using orth. polynomials

$$F_n(x) = \sum_{i=0}^{n} c_i \, \varphi_i(x)$$

then it can converge in L_2 norm to functions with jumps

$$y' = f(t,y)$$

$$\int_{t_n}^{t_{n+1}} \ldots \, dt$$

$$y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} f(t, y(t))\, dt$$

Apply Gauss method
need y at interpolation points
These will approximated too $\rightarrow$ RK method